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GENERAL RESEARCH

Boundary Layers in Rotating Flows

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ABSTRACT

A survey of work on rotating boundary layers on axisymmetric bodies is given. Exact similarity solutions are discussed and the limits for which the effects of transverse pressure gradient and curvature become predominant are noted. A general momentum-integral solution for both laminar and turbulent flow is presented, using a simplified form obtained from the generalization of similarity properties to include shear laws applicable in turbulent flow. The interaction of the boundary layer and the outer flow is discussed and solved for some examples of enclosed flows. Implications of such an interaction for certain aspects of the tornado problem are noted.

FIGURES

NOMENCLATURE

A	boundary-layer interaction parameter defined in Eq. (6-1)
a	stagnation-point inflow gradient
C	normalizing factor for the similarity velocities of Eq. (3-2)
c	constant in the shear law of Eq. (5-3a)
c_1, c_2	constants in the shear laws of Eqs. (5-5) and (5-6)
$\bar{c}_1$	iterated shear law parameter of Eq. (5-25)
f'	normalized meridional velocity profile in the boundary layer
g	normalized tangential velocity profile in the boundary layer
K	body thickness as defined in Eq. (3-1)
k_1, k_2	momentum-integral constants evaluated in Table 1
ℓ	characteristic axial length
$\dot{m}$	radial mass flow through the container of Figure 6 (positive for radial inflow)
n	boundary-layer coordinate normal to the wall
P	normalized pressure variation across the boundary layer
p	pressure
Q	boundary-layer volume flow divided by 2π
R	radius of the body or wall
r	radial coordinate
$\hat{r}$	defined in Eq. (6-4)
r^*	viscous core radius of the tornado
Re	radial Reynolds number, $\dot{m}/2\pi\ell\rho\nu$ (positive for radial inflow)

NOMENCLATURE (Continued)

s	boundary-layer coordinate along the wall
$\hat{U}$	maximum value of u_s
U_s	velocity component in the s direction outside the boundary layer
u	radial velocity
u_n	boundary-layer velocity component in the n direction
u_s	boundary-layer velocity component in the s direction (positive in the direction of increasing s)
V	tangential velocity outside the boundary layer
v	tangential velocity
w	axial velocity
x	similarity variable in the slender-body transformation of Eq. (4-2)
z	axial coordinate
α	exponent in the power-law variation of the body of revolution
β	exponent in the power-law variation of the velocity
Γ	vr outside the boundary layer
δ	boundary-layer thickness
ϵ	curvature parameter defined in Eq. (4-9)
ζ	$n/\delta(s)$
η	$\{[(R+n)/R]^2 - 1\}/\epsilon$
$\lambda_1, \lambda_2, \lambda_3$	profile constants given in Table 1
μ	exponent in the shear law of Eq. (5-4)
ν	kinematic viscosity

NOMENCLATURE (Continued)

ρ	density
σ	$(2 + \mu)/2(1 + \mu)$
τ_ϕ, τ_s	wall shear components
ψ	axisymmetric stream function
Ω	angular velocity
Subscript	
o	denotes value at $s = 0$

I. INTRODUCTION

The boundary-layer problems considered herein are for an axisymmetric steady flow of a viscous incompressible fluid, rotating with some circulation distribution which is not restricted a priori in any way, on walls which are also necessarily axisymmetric. Admittedly this generality leads to the inclusion of some artificial-looking cases, but the extra effort is justified by the clarification it provides for some limiting singular cases of practical interest.

Boundary layer problems in rotating flows are unique inasmuch as, for many applications, a strong interaction exists between the boundary layer and the outer flow. A method in boundary-layer theory, which has the flexibility and simplicity needed for the solution of such problems, will generally turn out to be an approximation involving some variation of the classical momentum-integral method. A critical evaluation of such a method involves comparison with known exact solutions, which in turn are always based on some similarity properties that hold for certain external conditions, at least in limited regions. The first part of this paper will be devoted to a review of similarity solutions together with discussion of the limitations of the flow regime in which the solutions are valid. Next, a momentum-integral method will be established for the laminar case in a form which is well suited for generalization to turbulent flow, and finally, applications to the interaction problem will be discussed.

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II. DIFFERENT TYPES OF ROTATING BOUNDARY LAYERS

In this section, different types of rotating boundary layers are delineated depending upon the orientation of the wall to the axis of rotation and the role of the driving pressure gradient. Because of the strongly preferential direction along the axis of rotation, walls which are circular cylinders parallel to the axis lead to flow problems quite distinct from those connected with nonparallel walls. The radius of the wall surface from the axis of rotation is called R. Three cases of R = const. are considered first, then R $\neq$ const. is discussed, and finally the precise distinction between these cases is determined by finding how nearly constant R must be for the R = const. cases.

The boundary-layer simplifications (valid if the boundary-layer thickness is much smaller than R) reduce the axisymmetric Navier-Stokes Equations for constant property fluids to the following set of equations, expressed in boundary-layer coordinates s and n, along and normal to the wall meridian (e.g., Ref. 1):

$$u_s \frac{\partial u_s}{\partial s} + u_n \frac{\partial u_s}{\partial n} - \frac{v^2}{R} \frac{dR}{ds} = -\frac{1}{\rho} \frac{\partial p}{\partial s} + \nu \frac{\partial^2 u_s}{\partial n^2} \tag{2-1}$$

$$\frac{u_s}{R} \frac{\partial vR}{\partial s} + u_n \frac{\partial v}{\partial n} = \nu \frac{\partial^2 v}{\partial n^2} \tag{2-2}$$

$$\frac{v^2}{R}\left[1 - \left(\frac{dR}{ds}\right)^2\right]^{1/2} = -\frac{1}{\rho} \frac{\partial p}{\partial n} \tag{2-3}$$

$$\frac{1}{R} \frac{\partial R u_s}{\partial s} + \frac{\partial u_n}{\partial n} = 0 \tag{2-4}$$

Here, u_s and u_n are the velocity components along s and n, v is the tangential velocity, p the pressure, ρ the density, and ν the kinematic viscosity.

If R is constant, then u_n and u_s are the radial velocity u and the axial velocity w, respectively, and s = z, which is the axial coordinate. It is seen that in this case Eq. (2-1) is completely decoupled from the circumferential flow unless $\partial p/\partial z$ depends on it. Boundary layers on circular cylinders parallel to the axis can be divided into three classes depending upon the role of $\partial p/\partial z$, as discussed in the following paragraphs.

Consider first the case where $\partial p/\partial z$ is independent of v. From Eqs. (2-1) and (2-3) it may be deduced that this will occur either if v is small in comparison with w or if v is independent of z.[1] Equation (2-1) then is the classical boundary-layer equation with the corresponding usual estimate of the boundary-layer thickness δ:

$$\delta \sim \sqrt{\frac{\nu z}{w}} \tag{2-5}$$

With v independent of z, Eq. (2-2) can be formally integrated to give

$$v = \text{const.} \int \left(\exp \int \frac{u}{\nu} \, dn \right) dn \tag{2-6}$$

To be consistent, it is seen that the radial velocity u must also be independent of z. Also, continuity requires that the axial velocity and consequently the axial pressure gradient be linear with z. In the special case when w vanishes, the possible solutions degenerate to the "suction" case with u constant and v given by Eq. (2-6).

[1] The family of the corresponding solutions without boundary-layer assumptions was investigated systematically by Donaldson (Ref. 2).

An example of this first type of boundary layer with a nonvanishing pressure gradient is the case of the superposition of stagnation-point flow with a constant line source to form a "stagnation circle" on a coaxial cylindrical wall. In this case the inviscid flow outside the boundary layer is given by

$$u = -a(r - R^2/r) \quad ; \quad w = 2az \tag{2-7}$$

In this example, Eq. (2-1) becomes identical with the boundary-layer equation associated with two-dimensional stagnation-point (Hiemenz) flow. Rotation of the cylinder may be superimposed without affecting u and w. The tangential velocity distribution in the boundary layer is then determined from Eq. (2-6).

Next, consider the second type of boundary layer in which the axial pressure gradient is coupled to the pressure gradient across the boundary layer. The terms in Eqs. (2-1) and (2-3) are of the same order if the orders of v and w are connected by the relation $w \sim v\sqrt{\delta/R}$. This relation together with the estimate of δ from Eq. (2-5) leads to

$$\delta \sim \left(\frac{\nu z}{vR^2}\right)^{2/5} R \quad ; \quad w \sim \left(\frac{\nu z}{vR^2}\right)^{1/5} v \tag{2-8}$$

This type of boundary layer was first considered by Stewartson (Ref. 3). He calculated similarity solutions for cases in which either a semi-infinite cylinder was rotating in a fluid otherwise at rest or a fluid was in solid-body rotation within a stationary cylinder, and also considered the shear layer at the interface of a rotating body of fluid bounded by a body of fluid at rest (Ref. 4). It should be noted that in this type of boundary layer the axial velocity w, although actually smaller than v (which provides the driving pressure gradient for w), is only smaller by a factor involving the fifth root of the Reynolds number.

Third, consider the case in which the boundary conditions force the axial velocity to be much less than the circumferential velocity. From Eqs. (2-1) and (2-3) the axial pressure gradient must then be of higher order (in an expansion parameter which turns out to be a Rossby number) than the transverse pressure gradient. This means that the leading term for v must be independent of z. Equation (2-6) is then valid again for v, with the attendant restriction that u must also be independent of z. Again, through continuity, w has to vary linearly with z. This approximation reduces the equations to the form considered in the first case, but their interpretation is now quite different. In the first case, u and w were completely independent of v, so that the flow pattern might have existed even if $v = 0$, whereas in the present case the secondary flow quantities u and w are strongly coupled to v through the higher order approximation of the rotation, which is z-dependent. This expansion procedure for small Rossby numbers has been systematically developed by Lewellen (Ref. 5). Actually, the boundary-layer approximation is only of incidental advantage for this method; no restriction on the characteristic radial dimension of the flow is required in its original formulation, which will be discussed further in connection with the boundary layer-rotating flow interaction problem in Section VI.

If the radius of the wall generator has an appreciable variation, the radial pressure gradient has a component which enters the meridional momentum equation (Eq. 2-1) and provides a coupling with the tangential momentum equation. This yields the fourth type of rotating boundary layer which is the most commonly considered case. Prandtl's classical assumption to neglect $\partial p/\partial n$ applies and leads to the elimination of Eq. (2-3) from the system of boundary-layer equations. This set may apply to either a rotating wall or a rotating external flow. However, by considering Eqs. (2-1) and (2-2) at the outer edge of the boundary layer, it may be established that a variation of the external circulation Γ with s cannot be assumed unless the external velocity U_s is zero. Therefore, $U_s \neq 0$ is possible only for a potential vortex (Γ = const.) or for $\Gamma = 0$ with rotating walls.

It now has to be established how parallel to the axis a wall may become before the normal pressure gradient, which is neglected in this fourth type of boundary layer, has to be reintroduced as comparable in its effect to the gradient parallel to the wall. With the pressure change across the boundary layer estimated from Eq. (2-3) to be of the order

$$\Delta p \sim \rho \frac{v^2}{R} \delta \tag{2-9}$$

the s-derivative of this pressure term is found to be negligible compared with the terms retained in Eq. (2-1) if both $\delta << R$ (which already has been assumed) and

$$\frac{d\delta}{ds} << \frac{dR}{ds} \tag{2-10}$$

From condition (2-10) it follows that the approximation leading to the elimination of Eq. (2-3) is applicable unless either $R =$ const. or the slope dR/ds falls below a small limit which itself is decreasing with the Reynolds number. Thus, the first three types of boundary layers considered must be restricted to cylinders for which R is quite constant with z.

The system of Eqs. (2-1), (2-2), and (2-4) applicable for most rotating boundary layers, will be treated by two methods: similarity transformations and approximate integral techniques.

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III. SIMILARITY SOLUTIONS

Similarity transformations have been investigated systematically for general three-dimensional boundary layers by Geis (Ref. 6). He has shown that a sufficient condition for similarity in the present problem is that the radius of the body of revolution varies as a power law of the meridional length[2]

$$R = Ks^{\alpha} \tag{3-1}$$

and that both circumferential and longitudinal velocities should vary with the same power of R. In conformity with this last condition let

$$v = CR^{\beta} g(\zeta) \quad ; \quad u_s = CR^{\beta} f'(\zeta) \tag{3-2}$$

where

$$\zeta = n/\delta(s) \quad ; \quad \delta = \sqrt{\frac{\nu s}{CR^{\beta}}} \tag{3-3}$$

Equations (2-1) and (2-2) can then be replaced by

$$f''' + \frac{1}{2}[1 + \alpha(2 + \beta)]ff'' - \alpha\beta f'^2 + \alpha g^2 = \alpha g^2(\infty) - \alpha\beta f'^2(\infty) \tag{3-4}$$

$$g'' + \frac{1}{2}[1 + \alpha(2 + \beta)]fg' - \alpha(\beta + 1)f'g = 0 \tag{3-5}$$

[2] Ishizawa (Ref. 7) has considered more general similarity conditions for boundary layers over bodies of revolution.

and from Eq. (2-4)

$$u_n = - \frac{CR^{\beta}\delta}{2s}[(\alpha\beta + 3)f + (\alpha\beta - 1)\zeta f'] \tag{3-6}$$

Unless $\beta = -1$, either $g(\infty)$ or $f'(\infty)$ must be zero, as was noted in the preceding section.

The wall shapes given by Eq. (3-1) have different character for different values of α. For $\alpha < 0$, the radius R_o approaches zero asymptotically; these infinite cusps are excluded from the present discussion. The cusped meridians connected with $\alpha > 1$ are also rather unrealistic. In the most interesting region of $0 < \alpha \leq 1$, similarity solutions are reached asymptotically for bodies behaving as $R \sim z^{\alpha}$ for large R. The value $\alpha = 0$ has to be excluded, as was shown before, together with bodies too slender for condition (2-10) to be satisfied, i.e., for bodies with $\alpha < \delta/r_o$.

Interesting conclusions can be drawn by comparing the growth laws of the body radius and of the boundary-layer thickness, i.e., Eqs. (3-1) and (3-3). If $\alpha(2 + \beta) < 1$, then δ grows with a higher power of s than R so that for some large value of s the boundary-layer approximations will break down. Conversely, for $1 > \alpha > 1/(2 + \beta)$, boundary-layer theory is applicable for half-infinite bodies without restrictions on the arclength s. In the limiting case, $\alpha(2 + \beta) = 1$, the growth of boundary layer and body are similar. It will be shown in Section IV that this case involves similarity between the first-order boundary-layer quantities and the terms which are usually retained for second-order effects. Correspondingly, a treatment of the problem is given which includes "higher-order" effects without resorting to an expansion procedure.

Within the class of similarity problems defined by Eqs. (3-1) to (3-5), the cases that have received the most attention in the literature correspond to $\alpha = \beta = 1$. For this value of α and β, Eqs. (3-4) and (3-5) were first used by von Kármán (Ref. 8) and by Cochran (Ref. 9) to find the flow due to an infinite rotating disk in an unbounded fluid at rest. It was later used by

Bödewadt (Ref. 10) to solve the opposite problem of a uniformly rotating fluid over an infinite stationary wall. Batchelor (Ref. 11) considered the family of solutions in which the ratio of the fluid rotation to the disk rotation varies from $-\infty$ to $+\infty$. The solutions for some of the flows discussed by Batchelor were obtained numerically by Rogers and Lance (Ref. 12). Stagnation flow against a rotating disk was studied by Hannah (Ref. 13) by including a source on the axis of rotation at infinity. Stuart (Ref. 14) and Nanda (Ref. 15) have given these flows a further degree of freedom by considering the disk to be porous with a uniform flow through it.

Values of β other than one are also of interest; the problem of a fluid rotating over a stationary wall with $\alpha = 1$ has been considered by King and Lewellen (Ref. 16) for $-1 \leq \beta \leq 1$, corresponding to the flow range between solid-body rotation and potential flow. It was found that the oscillations exhibited in the Bödewadt solution for $\beta = 1$ increased in magnitude and wavelength as β decreased; no solution exists for the potential flow $\beta = -1$.

The existence of oscillations in the profiles can be proven from the formal integral of Eq. (3-4). With the boundary conditions

$$f(o) = f'(o) = f'(\infty) = 0 \qquad (3\text{-}7)$$

Eq. (3-4) integrates twice to give

$$f^2(\infty) = \frac{-2}{1 + \alpha(2+\beta)} \int_0^\infty d\zeta \int_\zeta^\infty \{[1 + \alpha(2 + 3\beta)]f'^2 + 2\alpha[g^2(\infty) - g^2]\} d\zeta \qquad (3\text{-}8)$$

Thus, for $\alpha > 0$ and $\beta > -2$ it is always necessary to find $g > g(\infty)$ at some point within the boundary layer. The only time a monotonic profile is possible is when the wall rotates faster than the fluid. On the other hand, it can be seen from Eq. (3-5) that g must be monotonic when $\beta = -1$. This proves that there can be no solution for the potential vortex in which the fluid rotates faster than the wall (for all $\alpha > 0$), by contradiction with the established necessity of oscillations.

Physically, the oscillations occurring in the boundary layer when the fluid rotates faster than the wall appear to arise in the following manner. The radial inflow, induced by the retardation of the tangential velocity in the neighborhood of the wall, actually convects angular momentum into the region so strongly that it forces a local overshoot in the tangential velocity. This overshoot increases the centrifugal force locally to induce a radial outflow, which convects a circulation defect, and the whole process is repeated to yield the oscillatory approach to infinity.

For $\alpha \neq 1$, apparently the only numerical solutions in the published literature are the computations of Geis for $\beta = 1$, and $\alpha = 4/5$ and $3/2$ for rotating walls with the fluid at rest. The profile shapes of these solutions are very similar to those obtained for $\alpha = 1$.

Investigation of the meridional mass flow carried in the boundary layer shows interesting properties of the similarity solutions found thus far. From Eqs. (3-2) to (3-3), it follows that

$$Q = \int_0^\delta u_s R \, dn = \left[C \nu R^{2+\beta+1/\alpha} K^{-1/\alpha} \right]^{1/2} f(\infty) \tag{3-9}$$

First, we note that the geometric parameter K of Eq. (3-1), which did not enter Eqs. (3-4) and (3-5), now explicitly scales the flux. For instance, all disk flow results ($\alpha = 1$, $K = 1$) are directly applicable to cones with $K < 1$; Eq. (3-9) shows that the flux increases with decreasing cone angle for the same radius. This shows that the effect of the reduced centrifugal force component is more than compensated by the increased wall area.

Of particular interest is the sign of Q and its derivative with respect to R. For $0 < \alpha \leq 1$ and $\beta > -2$, the exponent of R in Eq. (3-9) is always positive. (Actually, $\beta \geq -1$ is a necessary condition for a stable rotating outer flow.) For all known similarity solutions of this group, the boundary layer takes in fluid and flow is radially outward if the wall rotates and the fluid is at rest; for rotating fluids over walls at rest, the radial boundary-layer flow is directed inward and fluid is expelled into the outer flow. (No

mathematical proof that there are no exceptions to this rule has been found, but physically their occurrence is improbable.)

The latter case is anomalous inasmuch as the fluid first has to flow into the boundary layer in the course of its development; in the case of similar solutions, the inflow is thought of as occurring at infinity, and the boundary layer begins with an infinite reservoir of Q which is being steadily expelled as the radius decreases. These solutions may be thought of as having "terminal" similarity. Although, as pointed out by Stewartson (Ref. 3), there is no assurance for an asymptotic approach to this group of similarity solutions if the actual boundary-layer development begins at a finite radius, conditions similar to those of the similarity solutions are approached near the center of a finite disk. Mack (Ref. 17) argues that the Bödewadt solution is approached at the center of a finite stationary disk perpendicular to the axis of a fluid rotating as a solid body, and King and Lewellen (Ref. 16) speculate that the same approach to a similarity solution holds for more general values of β.

Investigation of the initial development of the boundary layer at a finite radius leads to a different category of similarity solutions. This has been done by Stewartson (Ref. 3) for a finite stationary disk in a rotating flow. Based on this initial solution, Mack (Ref. 18) calculated a series expansion of the Blasius type for boundary layers on stationary disks in external flows having power-law variations of the circumferential velocity, ranging from potential vortex to rigid rotation. His "exact" series solution served as a test case for the momentum methods to be discussed presently. The initial similarity properties (power laws) of Stewartson's solution also play an important role in the analysis using momentum methods and will be obtained later in that context.

Mack undertook a second generalization of Stewartson's work by calculating the initial similarity profiles for the case in which both the disk and the fluid rotate (Ref. 19). If the two rotational speeds have only a small difference, the problem can be linearized; the solution approaches the well-known Ekman spiral (Ref. 20), which is obtained in all limits of this kind (e.g., also by Rogers and Lance).

The growth and decay of the boundary layer may be analyzed more completely with the aid of integral methods, but first let us consider the extension of similarity solutions to include curvature effects.

IV. CURVATURE EFFECTS

As already noted, the application of the system of Eqs. (2-1), (2-2), and (2-4) to very slender bodies is limited by the fact that when the boundary-layer thickness becomes of the same order as the radius of the body it is necessary to include the transverse pressure gradient. It can also be seen that this is the regime in which

$$\frac{1}{r}\frac{\partial}{\partial n} = O\,\frac{\partial^2}{\partial n^2} \tag{4-1}$$

and can no longer be neglected in the viscous terms. It should perhaps be repeated that the transverse pressure gradient can become larger than the parallel component without curvature terms becoming important. This is Stewartson's case for which the body radius is much larger than the boundary-layer thickness and the body surface is quite parallel to the axis. Of course, even in Stewartson's problem if the body is long enough an axial position where curvature terms are important will be reached since the boundary-layer thickness increases with length while the body radius must remain essentially constant.

In the curvature regime it is appropriate to replace the boundary-layer approximation with a slender-body approximation which assumes that characteristic radial dimensions are much smaller than any characteristic axial dimension. Lewellen (Ref. 21) has shown that the resulting slender-body equations are compatible with a similarity transformation of the form

$$\begin{aligned} v &= \frac{z^m}{x}\, g(x) \\ w &= z^m f'(x) \\ x &= r z^{(m-1)/2} \end{aligned} \tag{4-2}$$

This is similar to the transformation of Eqs. (3-2) to (3-5) when δ and R vary as the same power law of s, and s is taken equal to z in agreement with the slender-body approximation. Thus, whenever

$$\alpha = \frac{1}{\beta + 2} \tag{4-3}$$

Equations (3-4) and (3-5) can be extended to include the effects of curvature. For this case, the equations are

$$(1 + \epsilon\eta)f''' + \epsilon f'' + ff'' - (1 - 2\alpha)f'^2 + \frac{\alpha g^2}{1 + \epsilon\eta}$$

$$= \alpha g^2(\infty) + (1 - 2\alpha)\epsilon P - (1 - 2\alpha)f'^2(\infty) \tag{4-4}$$

$$(1 + \epsilon\eta)g'' + fg' - (1 - \alpha)f' g = 0 \tag{4-5}$$

$$P' = g^2(1 + \epsilon\eta)^{-2} \tag{4-6}$$

The new definitions of f and g are

$$v = CR^{\beta}\left(\frac{R}{r}\right)g(\eta) \tag{4-7}$$

$$u_s = CR^{\beta}f'(\eta) \tag{4-8}$$

with the variable ζ replaced by

$$\eta = \frac{1}{\epsilon}\left[\frac{(R + n)^2}{R^2} - 1\right] \quad ; \quad \epsilon = 2\sqrt{\frac{\nu}{CK^{2+\beta}}} \tag{4-9}$$

and the new independent function P defined by

$$p = \frac{1}{2}\rho(CR^{\beta})^2 \epsilon P(\eta) \tag{4-10}$$

The parameter ϵ is a measure of the effect of curvature and is actually equal to $2\delta/R$ when δ is small. For $\epsilon \to 0$, $\eta \to \zeta$ and Eqs. (4-2) to (4-6) reduce to Eqs. (3-4) and (3-5) when Eq. (4-3) is satisfied.

The body shape determined by Eq. (4-3) for uniform rotation, i.e., when $\beta = 1$, is $\alpha = 1/3$. The velocity profiles in the boundary layer of such a 1/3-power-law body which rotates in a fluid at rest may be obtained as a function of ϵ by numerical integration of Eqs. (4-4) to (4-6) with the boundary conditions

$$\begin{aligned} \eta = 0:&\quad f = f' = 0 \quad , \quad g = 1 \\ \eta = \infty:&\quad f' \to 0 \quad , \quad g \to 0 \quad , \quad P \to 0 \end{aligned} \tag{4-11}$$

The results of this computation[3] are shown in Figures 1 to 3 for a range of values of ϵ. The profiles obtained for the rotating disk boundary layer and the square-root power-law body are also included for comparison. Increasing the curvature effect is qualitatively similar to decreasing the power-law exponent of the body.

[3]These computations were performed by the Aerospace Computation and Data Processing Center. The authors are particularly indebted to R. F. Kramer for this work.

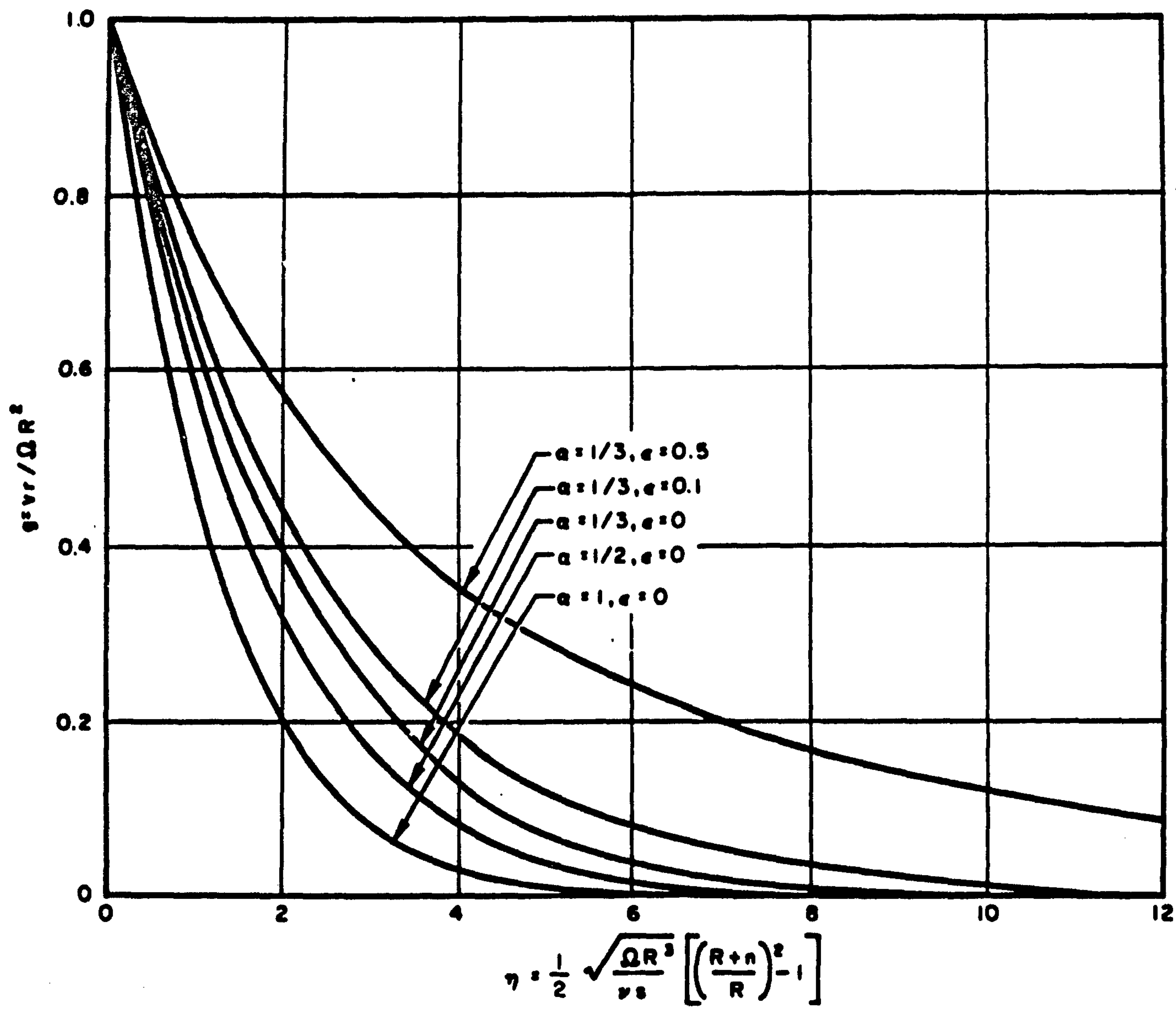

Figure 1. Tangential Velocity Profiles in the Boundary Layer of a Rotating Power-Law Body ($R \sim s^{a}$) Including the Effects of Curvature ϵ

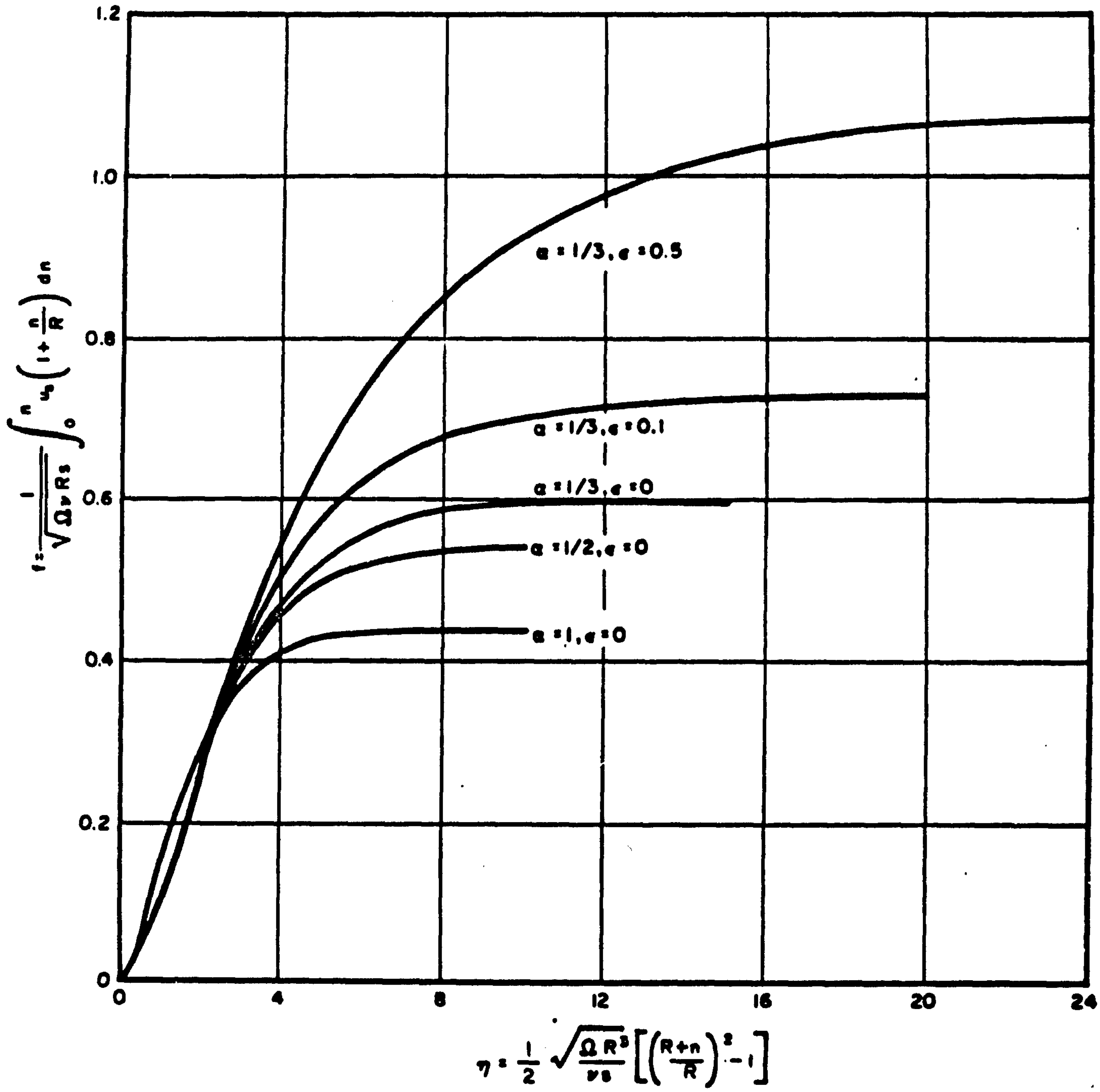

Figure 2. Meridional Volume Flow in the Boundary Layer of a Rotating Power-Law Body ($R \sim s^{a}$) Including the Effects of Curvature ϵ

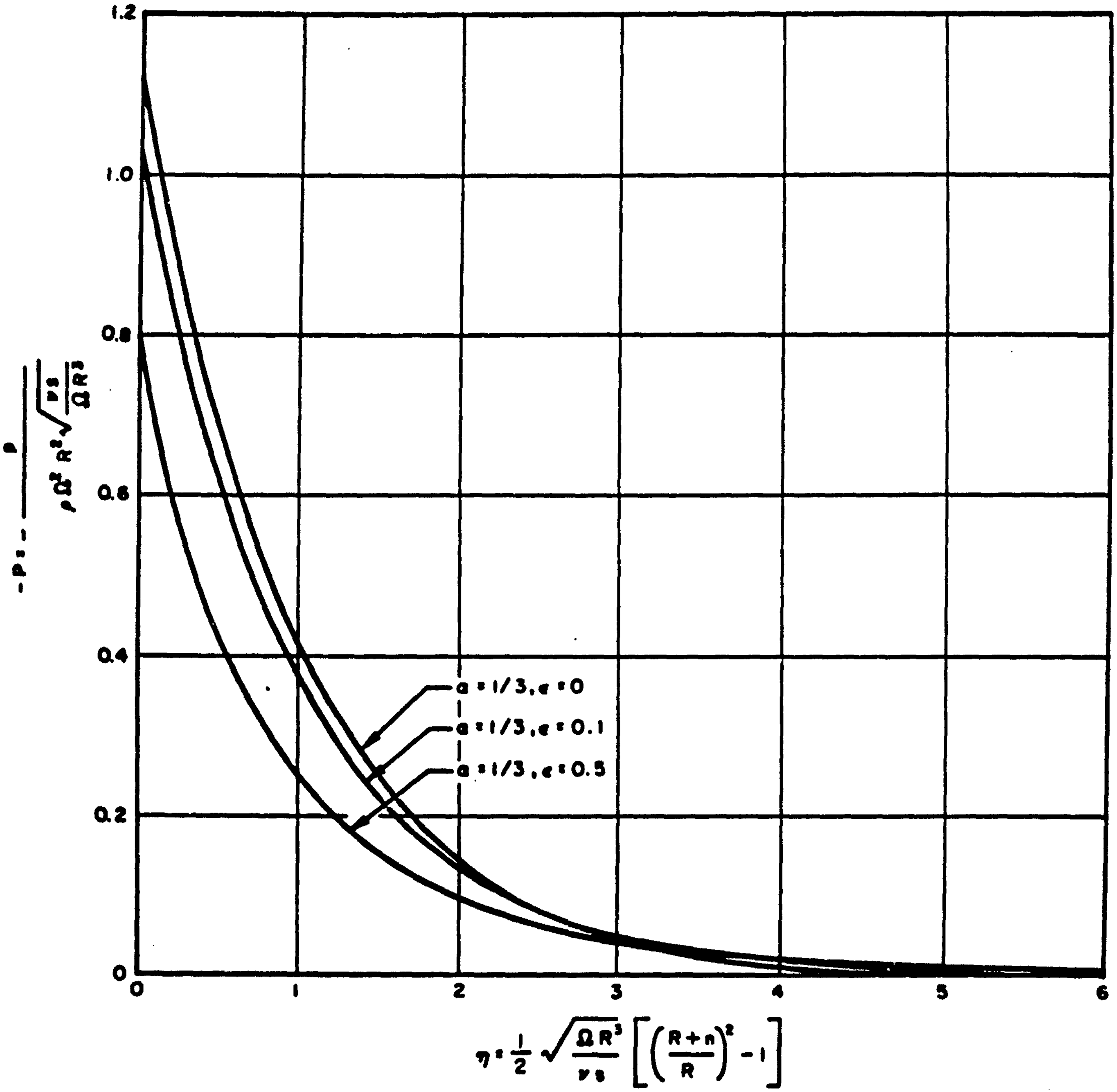

Figure 3. Pressure Distribution across the Boundary Layer of a Rotating Power-Law Body ($R \sim s^{a}$) Including the Effects of Curvature ϵ

V. MOMENTUM INTEGRAL SOLUTION FOR A ROTATING FLUID OVER A STATIONARY WALL

The first treatment of the boundary-layer problem in rotating flows by momentum-integral methods was given by von Kármán (Ref. 8), who considered the rotating disk, in both the laminar and turbulent cases. Taylor (Ref. 22) calculated the laminar boundary-layer development on a wall at rest, with a vortex as the outer flow. In both these investigations the circumferential and radial momentum-integral equations were used to determine the thickness and the radial velocity peak of the boundary layer. Von Kármán's results (in the laminar case) were subsequently confirmed by Cochran's exact similarity solution. In extending Taylor's work, Cooke (Ref. 23) introduced a further refinement, admitting two different thicknesses for the radial and the tangential layer, and requiring that the radial boundary-layer equation at the wall also be fulfilled (referred to herein as the "wall condition"). Cooke's results were quite different from Taylor's. Therefore, although the application of momentum-integral methods to the boundary-layer problems of a rotating wall in a fluid at rest appears to be straightforward, the opposite case of a rotating fluid confined by a stationary wall appears more complicated. This latter problem is the one to be carried through in detail here.

A critical evaluation of the momentum-integral methods became possible with the help of Mack's series (Ref. 18) [which superseded earlier solutions by Garbsch (Ref. 24) who used a "local similarity" type of approach]. Discussions of different integral methods were carried out by Mack (Ref. 17, 25) and by King (Ref. 26). In this section, a method will be presented which incorporates the best recommendations of both authors for the laminar case and which permits generalization to the case of turbulent flow.

The analysis is based on the tangential and meridional momentum-integral equations, which can be derived from Eqs. (2-2) and (2-1) and are written in the following form, with $U_s = 0$:

$$\frac{d}{ds}(Q\Gamma) - \lambda_1 Q \frac{d\Gamma}{ds} = \lambda_1 \frac{R^2 \tau_\phi}{\rho} \tag{5-1}$$

$$\lambda_2 \frac{d}{ds}\left(\frac{Q^2}{R\delta}\right) + \lambda_3 \frac{dR}{ds} \frac{\Gamma^2 \delta}{R^2} = - \frac{R\tau_s}{\rho} \tag{5-2}$$

where λ_1, λ_2, and λ_3 are profile parameters defined in Table 1. By use of the appropriate profile shapes (also given in Table 1) and shear laws, Eqs. (5-1) and (5-2) are applicable to both laminar and turbulent flow; they determine the two unknowns, namely, the boundary-layer thickness δ and the meridional boundary-layer flux Q (defined in Eq. 3-9), with $\Gamma(s)$ and $R(s)$ (body shape) given.

For the shear in the turbulent case, von Kármán's adaptation of the Blasius law to the three-dimensional case (Ref. 8) will be accepted:

$$\tau_{res} = c\rho(\hat{U}^2 + V^2)\left(\frac{\nu}{\sqrt{\hat{U}^2 + V^2}\,\delta}\right)^{\mu} \tag{5-3a}$$

$$\tau_\phi = \tau_{res} \frac{V}{\sqrt{\hat{U}^2 + V^2}} \quad ; \quad \tau_s = \tau_{res} \frac{\hat{U}}{\sqrt{U^2 + V^2}} \tag{5-3b}$$

where $\mu = 1/4$, after Blasius, and $c = 0.0225$; $\hat{U} = (u_s)_{max}$. Numerical calculations based on the set of equations given thus far were first carried out for the turbulent case by Weber (Ref. 27).

Table 1. Momentum-Integral Profiles and Constants

Profiles and Constants	Laminar ($\mu = 1,\ \sigma = 3/4$)	Turbulent ($\mu = 1/4,\ \sigma = 9/10$)
$u_s/\hat{U} = f'(n/\delta)$	$27/4\ \zeta(1-\zeta)^2$	$1.69\ \zeta^{1/7}(1-\zeta)^2$
$v/V = g(n/\delta)$	$2\zeta - \zeta^2$	$\zeta^{1/7}$
$\lambda_1 = f(1)/\int_0^1 f'(1-g)\,d\zeta$	2.5	4.93
$\lambda_2 = \int_0^1 f'^2 d\zeta / f^2(1)$	1.375	1.63
$\lambda_3 = \int_0^1 (1-g^2)\,d\zeta$	0.467	0.222
c_1	$g(0) = 2$	0.0225
c_2	$f''(0)/f(1) = 12$	$c_1/f(1) = 0.0513$
$k_1 = \frac{3+2\mu}{2+\mu}\frac{\lambda_2}{\lambda_3} + \frac{c_2}{c_1}\frac{1}{\lambda_1\lambda_3}$	10.05	13.5
$k_2 = \sigma^{-\sigma}(c_1\lambda_1)^{2\sigma-1}k_1^{\sigma-1}$	1.56	0.145

As a first step towards a simplified practical method, the "starting" properties of the boundary-layer development will be investigated analytically, based on Eqs. (5-1) and (5-2). It is assumed that at radius $R = R_o$ the boundary layer begins with zero thickness δ and with vanishing meridional velocity u_s (and Q).[4] In the laminar case, such an analysis should recover

[4] For a discussion of the effect of finite initial values, see Mack (Ref. 17).

the similarity properties of Stewartson's solution (Ref. 3) as far as the power-laws of mutual interdependence are concerned; the constants of proportionality, however, as derived from the momentum-integral equations, will not agree with Stewartson's exact similarity solutions. unless, naturally, Stewartson's profiles are used to calculate the different profile parameters in Eqs. (5-1) and (5-2).

The aim in the turbulent case, then, is to find the proper extension of Stewartson's "initial" similarity, which will differ from the laminar case primarily (i. e., in the power-law exponents) because of the difference in the shear law. The turbulent shear formulas (5-3a) and (5-3b) can be simplified for the initial flow situation, where $\hat{U} \ll V$; it is found that

$$\tau_\phi = c\rho V^2 \left(\frac{\nu}{V\delta}\right)^\mu \quad ; \quad \tau_s = c\rho \hat{U} V \left(\frac{\nu}{V\delta}\right)^\mu \tag{5-4}$$

or, using the variables Γ and Q,

$$\tau_\phi = c_1 \rho \frac{\Gamma^2}{R^2} \left(\frac{\nu R}{\Gamma \delta}\right)^\mu \tag{5-5}$$

$$\tau_\phi = c_2 \rho \frac{Q\Gamma}{\delta R^2} \left(\frac{\nu R}{\Gamma \delta}\right)^\mu \tag{5-6}$$

Note that these initial shear laws cover both the laminar and the turbulent cases for the proper choice of the exponent μ. The expressions for laminar flow are found for $\mu = 1$; turbulent cases are obtained by using the Blasius value $\mu = 1/4$, or some other empirical value, e. g., $\mu = 0$ for rough surfaces. The appropriate definitions of c_1 and c_2 are given in Table 1. In the laminar case, these constants depend on the gradient of the velocity profiles at the wall.

To obtain the initial similarity solutions, Eqs. (5-1), (5-2), (5-5), and (5-6) are solved inserting the constant values $R = R_o$, $\Gamma = \Gamma_o$, and $dR/ds = R'_o$. Put $\delta = BQ^{\kappa}$ and eliminate dQ/ds between Eqs. (5-1) and (5-2), noting that $Q(0) = 0$; B and κ are thus determined. The result of this somewhat lengthy but straightforward calculation is

$$\delta^{2+\mu} = c_1 \lambda_1 k_1 \nu^{\mu} R_o^{1+\mu} \Gamma_o^{-(1+\mu)} (-R'_o)^{-1} Q \tag{5-7}$$

where

$$k_1 = \frac{3 + 2\mu}{2 + \mu} \frac{\lambda_2}{\lambda_3} + \frac{c_2}{c_1} \frac{1}{\lambda_1 \lambda_3} \tag{5-8}$$

It is reassuring to note that, indeed, the exponents in this relation contain μ only, and that for $\mu = 1$ Stewartson's similarity law (up to the definition of the constant k_1) is re-established.

Thus far, the results obtained are consistent with the original methods proposed by Taylor (Ref. 22) and Weber (Ref. 27) which are easily manageable with modern computing techniques. Nevertheless, further simplifications will be sought, which will make an analytical solution possible. This is particularly desirable in view of the fact that it can be done with little or no loss in accuracy.

The method proposed is to use Eq. (5-7) not only initially, but throughout the whole range of the calculations, with the initial constants R_o, Γ_o, and R'_o replaced by the variable local values R, Γ, and R'. This relation will replace the meridional momentum equation (5-2). The main argument for use of the generalized relation (5-7) is that in conjunction with the tangential momentum equation (5-1), it yields not only the proper initial similarity, but also the terminal similarity properties of the solutions of King and Lewellen discussed in Section III. It can be shown that such a relation is the only

possible power law connecting Q and δ that is consistent with both initial and terminal similarity. Before considering the generalization of the terminal similarity to the turbulent case, a simple interpretation of Eq. (5-7) for laminar flow will be discussed.

A variation of the momentum-integral method proposed by Anderson (Ref. 28) and by Mack (Ref. 25) replaces the meridional momentum-integral equation by the wall condition of the meridional laminar boundary-layer equation, which relates the pressure gradient to the normal shear gradient. The corresponding relation has the same form as the generalized Eq. (5-7), although the constant k_1 has a value slightly different from the result of Eq. (5-8) (given in Table 1). Comparison of the results of this method with solutions provided by the original Taylor scheme was carried out by Mack (Ref. 25) and by King (Ref. 26) and have shown that the agreement is close everywhere; the advantage of the use of Eq. (5-7) lies in its analytical simplicity.

For turbulent flow, the interpretation of the proposed method by use of the wall condition is not possible. Moreover, the shear laws given by Eqs. (5-5) and (5-6), which were needed to obtain Eq. (5-7), can be used for laminar flow ($\mu = 1$) over the whole range of the calculations, but they are only initial laws for turbulent shear. However, Eq. (5-4) gives values of the shear which differ from the results of von Kármán's original formulas (5-3b) only by a factor $(1 + \hat{U}^2/V^2)^{(1-\mu)/2}$, and $\hat{U}$ V can be considered as an upper limit for the radial velocity peak. While an iterative correction for such a discrepancy could easily be devised, no such steps will be taken, as turbulent shear laws in three-dimensional flow situations have uncertainties beyond the factor quoted. The somewhat lower values provided by Eqs. (5-5) and (5-6) are accepted, for the sake of analytical simplicity. Comparison with results of Weber (Ref. 27), who uses Eq. (5-3b), indicates that the effects of the simplification on the final results are not significant.

The task of determining the power-law properties of the terminal similarity solutions, as generalized to the turbulent case, can now be undertaken by various methods involving the momentum-integral equations. It can

be shown that for all shear laws considered thus far, and for all values of μ, terminal similarity solutions are obtained for which $\hat{U}$ and V vary with the same power of R, i.e., $\hat{U} \sim V \sim R^{\beta}$. The factor between the different turbulent shear law approximations noted in the previous paragraph has therefore a constant value and does not affect the powers of the terminal similarity. Furthermore, it can be shown that the same power laws are obtained with Taylor's and Weber's method, i.e., Eqs. (5-1) and (5-2), as with the simplified approach proposed here based on Eq. (5-5) and the generalized Eq. (5-7). The latter equations are repeated here for convenience:

$$\frac{d}{ds}(Q\Gamma) - \lambda_1 Q\frac{d\Gamma}{ds} = \lambda_1 c_1 \Gamma^2\left(\frac{\nu R}{\Gamma\delta}\right)^{\mu} \qquad (5\text{-}9)$$

$$\delta^{2+\mu} = c_1 k_1 \lambda_1 \nu^{\mu} R^{1+\mu} \Gamma^{-(1+\mu)} (-R')^{-1} Q \qquad (5\text{-}10)$$

In terms of the boundary-layer variables, it may be noted that $\hat{U} \sim V$ implies the relation $Q \sim \Gamma\delta$. The terminal similarity dependence of Q and δ can be expressed as follows: if

$$\Gamma = CR^{1+\beta} \qquad (5\text{-}11)$$

then

$$Q = B_1 \nu^{\mu/(1+\mu)} R(-R')^{-1/(1+\mu)} \Gamma^{1/(1+\mu)} \qquad (5\text{-}12)$$

$$\delta = B_2 \nu^{\mu/(1+\mu)} R(-R')^{-1/(1+\mu)} \Gamma^{-\mu/(1+\mu)} \qquad (5\text{-}13)$$

The generalization of Bödewadt's similarity to the turbulent case is of particular interest; the axial velocity derived from Eqs. (5-11) and (5-12) for $\beta = 1$ and $R' = -1$ is

$$w = \frac{1}{R}\,\frac{dQ}{dR} = \frac{3+\mu}{1+\mu}\,B_1 C^{1/(1+\mu)}\nu^{\mu/(1+\mu)}R^{(1-\mu)/(1+\mu)} \tag{5-14}$$

so that for $\mu = 1/4$, $w \sim R^{3/5}$.

The constants B_1 and B_2 depend on the choice of the particular integral method; based on Eqs. (5-9) and (5-10), it is found for the disk ($R' = -1$) that

$$B_1 = (c_1\lambda_1)^{1/(1+\mu)}k_1^{-\mu/2(1+\mu)}N^{-2(1+\mu)/(2+\mu)} \tag{5-15}$$

$$B_2 = (c_1\lambda_1 k_1 B_1)^{1/(2+\mu)} \tag{5-16}$$

where

$$N = \left(\lambda_1 - \frac{2+\mu}{1+\mu}\right)(1+\beta) - 1 \tag{5-17}$$

The agreement of the numerical values obtained from these equations with the numbers calculated by King and Lewellen is poor but reasonable in view of the big difference between the smooth profiles used for the momentum methods and the oscillatory similarity solutions. For instance, in the Bödewadt case ($\beta = 1$; $\mu = 1$), Eq. (5-15) yields $B_1 = 1.26$, while the exact value is 0.68.

Actually, there even exists a limit for a meaningful solution, as the value of N after Eq. (5-16) must be positive. The limiting exponent β which makes N vanish can have no true significance as it depends on the profile parameter λ_1 (as well as on other possible variations of the present method). It can be noted, however, that in the potential flow case, i.e., $\beta = -1$, neither

the King-Lewellen similarity solution nor the approximate solution exists. The spurious limit of the approximate method naturally can shed no light on the difficult existence questions involving terminal similarity, but it is consistent with the established nonexistence for $\beta = -1$.

For the case of fluids rotating over a fixed wall of finite extent, the significance of the King-Lewellen similarity solutions and their turbulent generalizations lies in the possibility that they may be approached as terminal solutions for $R \to 0$. As the different variants of the momentum methods give very similar results in the terminal phase, and identical solutions in the initial development, it is not surprising that their mutual agreement is strong. Thus, the choice of Eqs. (5-9) and (5-10) can be justified by the fact that they are equally as accurate as Taylor's method and lead to the following closed-form solutions:

$$Q = k_2 \nu^{2(1-\sigma)} \Gamma^{\lambda_1 - 1} \left\{ \int_0^s \Gamma^{2-\lambda_1/\sigma} [R(-R')]^{(1-\sigma)/\sigma} ds \right\}^{\sigma}$$

$$\sigma = \frac{2+\mu}{2(1+\mu)} \quad ; \quad k_2 = \sigma^{-\sigma} (c_1 \lambda_1)^{2\sigma - 1} k_1^{\sigma - 1} \tag{5-18}$$

and the value of δ is given by

$$\delta = \left(\frac{c_1^2 \lambda_1^2 k_1}{\sigma} \right)^{(2\sigma-1)/2} \nu^{2(1-\sigma)} \Gamma^{\lambda_1(1-1/2\sigma)-1} R^{1/2\sigma} (-R')^{(1/2\sigma)-1}$$

$$\times \left\{ \int_0^s \Gamma^{2-\lambda_1/\sigma} [-RR']^{(1-\sigma)/\sigma} ds \right\}^{(2\sigma-1)/2} \tag{5-19}$$

The remaining task is to make a choice of profile shapes and to compute the constants needed in these formulas. In Table 1, the most widely used profile shapes are quoted, together with the numbers derived from them.

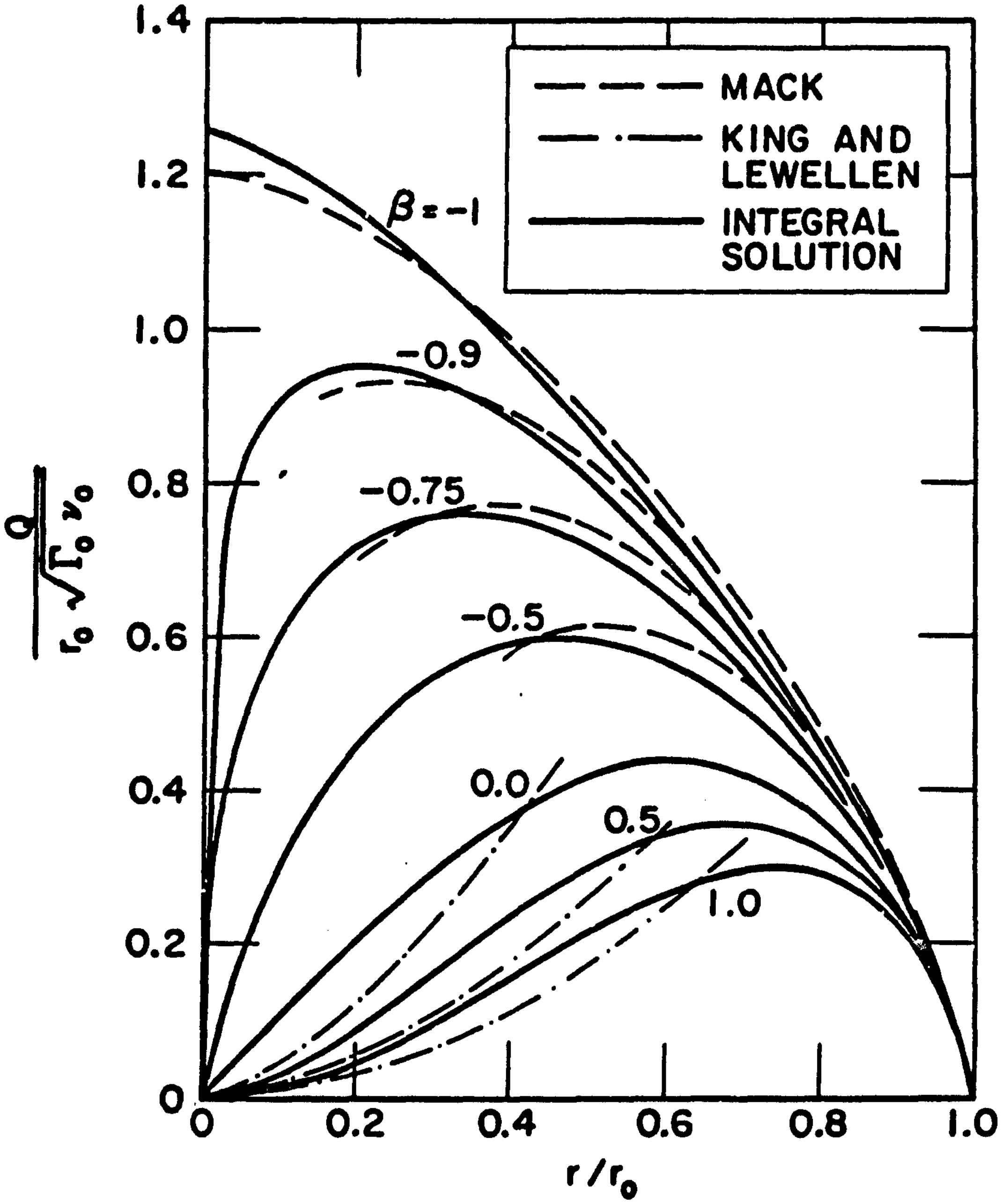

Figure 4. Radial Volume Flow in the Laminar Boundary Layer of a Finite Stationary Disk which is Perpendicular to a Fluid Rotating with $\Gamma \sim r^{\beta+1}$

As a first example, the results of Eq. (5-18) for the laminar case with $\Gamma = cR^{1+\beta}$ over a finite disk are compared to Mack's series solution (Ref. 18). Figure 4 shows that Q, according to Mack's results, grows monotonically inward for a potential vortex $\beta = -1$; the initial development for $\beta > -1$ is described by the same (Stewartson) solution, but a maximum in Q develops which, according to Mack, nearly coincides with the practical convergence limit of his series. The results of the momentum method proposed in Eq. (5-18) closely reproduce the position and the magnitude of the extremum of Q, including the monotonic growth for $\beta = -1$. Also plotted in Figure 4 are the predictions of some similarity solutions of King and Lewellen; they are approached by the momentum method results, within the limits discussed previously.

Next, consider the potential vortex, Γ = const., for different shear laws. For a flat finite disk, $s = R_o - R$ and Eqs. (5-18) and (5-19) give

$$Q = (c_1\lambda_1)^{2\sigma-1} k_1^{\sigma-1} \nu^{2(1-\sigma)} \Gamma^{2\sigma-1} R_o \left[1 - \left(\frac{R_o - s}{R_o}\right)^{1/\sigma}\right]^{\sigma} \qquad (5\text{-}20)$$

$$\delta = \left(c_1^2\lambda_1^2 k_1\right)^{(2\sigma-1)/2} \nu^{2(1-\sigma)} \Gamma^{2(\sigma-1)} R_o \left[1 - \left(\frac{R_o - s}{R_o}\right)^{1/\sigma}\right]^{(2\sigma-1)/2} \left(\frac{R_o - s}{R_o}\right)^{1/2\sigma}$$

(5-21)

For small s, $Q \sim (s/R_o)^{\sigma}$ and $\delta \sim (s/R_o)^{(\sigma-1)/2}$. This is in agreement with Stewartson's similarity for laminar flow ($\sigma = 3/4$) and generalizes it for the turbulent case.

If $\mu = 0$ and $\sigma = 1$, Eqs. (5-18) and (5-19) simplify to

$$Q = c_1\lambda_1\Gamma^{\lambda_1 - 1} \int_0^s \Gamma^{2-\lambda_1}\, ds \qquad (5\text{-}22)$$

$$\delta = c_1\lambda_1 k_1^{1/2} R^{1/2}(-R')^{-1/2}\Gamma^{(\lambda_1/2)-1}\left\{\int_o^s \Gamma^{2-\lambda_1}\, ds\right\}^{1/2} \tag{5-23}$$

The result, appropriate for turbulent flow with rough surfaces, may also be used approximately for smooth surfaces ($\mu = 1/4$) if c_1 is replaced by a value $\overline{c}_1$, estimated iteratively as an "average" friction coefficient from a Blasius-type empirical shear law. In the case of constant (or nearly constant) values of Γ, Eq. (5-12) gives

$$\delta \approx c_1\lambda_1\left(\frac{k_1 Rs}{-R'}\right)^{1/2} \tag{5-24}$$

and according to Eq. (5-2), $\overline{c}_1$ has to be defined as

$$\overline{c}_1 = c_1\left(\frac{\nu R}{\Gamma \delta}\right)^{\mu} \tag{5-25}$$

From Eqs. (5-24) and (5-25), $\overline{c}_1$ can be evaluated; it varies with the power $\mu/2(1 + \mu) = 1/10$ of R/s. Evaluated for a finite disk at the midpoint $R = s$, it is found that

$$\lambda_1 c_{1_{R=s}} = (\lambda_1 c_1)^{1/(1+\mu)} k_1^{-\mu/2(1+\mu)} (\nu/\Gamma)^{\mu/(1+\mu)} = 0.133(\nu/\Gamma)^{1/5} \tag{5-26}$$

Equation (5-22), with $\lambda_1 c_1$, replaced by $\lambda_1\overline{c}_1$ after Eq. (5-26), was used for the interaction problem in Ref. 34. (The original number given by Rott (Ref. 29) was 0.135, based on the assumption $c_2 = 0$.)

The discussions of this section can provide an explanation, in retrospect, for why methods with an additional variable like Cooke's (Ref. 23) may lead to an erroneous result rather than an improvement. Briefly, Cooke's additional parameter (the boundary-layer thickness ratio) can behave, for $R \rightarrow 0$, as R^{γ}, where any value of $\gamma \neq 0$ admits a solution which is in disagreement with the admissible terminal similarity. The investigations of King (Ref. 26) [who also corrected a numerical mistake in Cooke's analysis] indicate that indeed a singular behavior of the additional Cooke parameter occurs for $R \rightarrow 0$; a different three-parameter solution investigated by King apparently suffered a similar fate. It seems that the three equations, with partly overlapping physical content, do not provide enough independent information for use in such an approximate method to adequately control the additional degree of freedom. It is believed that an improvement of the present method must account primarily for the change in character of the profile, which begins smoothly for growing Q but probably always acquires some oscillatory character for outflow.

Naturally, a breakdown of the boundary-layer theory always occurs as $R \rightarrow 0$; this is outside of the scope of this paper. It has to be noted that such a breakdown can also occur at a finite radius due to a local disturbance; observations by Rosenzweig, Ross, and Lewellen (Ref. 30) have shown that a vortex chamber, a large fraction of the fluid contained in the boundary layer approaching an exit hole is actually re-ejected axially into the chamber, instead of leaving through the hole. A quantitative theory of this effect is not yet available, but it can be stated that the type of breakdown in the boundary-layer development, which in plane flow is described by the term "separation", will lead in rotating fluids to effects which extend very far axially outside the boundary layer.

In summary, a momentum method is proposed which is believed to provide a tool useful in both laminar and turbulent flow, including somewhat unusual regions of outflow from the boundary layer, but excluding zones of very local but violent flow changes.

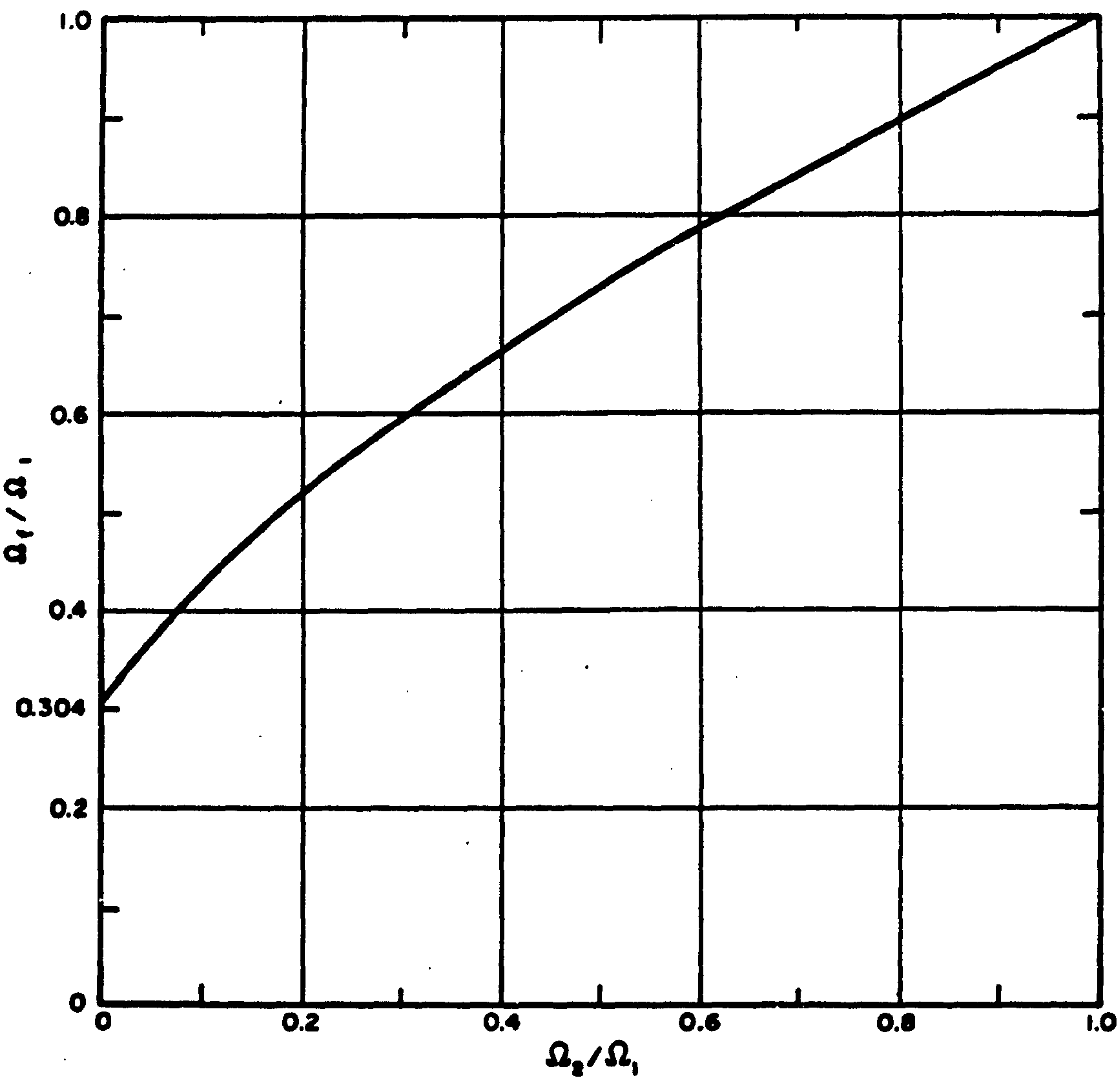

Figure 5. **Angular Velocity Ω_f of the Main Body of Fluid Between Two Infinite Rotating Disks as a Function of the Ratio of the Angular Velocity of the Two Disks**

VI. APPLICATIONS

For most applications of rotating boundary-layer theory, the finite boundaries of the external flow make it necessary to consider the mutual interaction of the boundary layer and the external flow. Flow about a rotating disk depends upon the size and shape of the container within which the disk rotates. Even when the container is large, it is necessary to consider the source and disposal of the flow induced in the rotating boundary layer in order to ascertain the eventual influence of the boundary-layer flow on the primary flow. Some examples where this interaction problem has been solved will now be considered, and then speculation offered as to how this interaction may be important in the case of a tornado.

Perhaps the simplest example of the interaction of the boundary layer and the external flow is that of flow between two infinite disks discussed by Batchelor (Ref. 11) and Stewartson (Ref. 31). If both disks rotate at the same rate, the total fluid between the disks will also rotate at this rate and there will be no boundary layer. However, for the general case, when the disks rotate at different rates, the fluid between the disks must adjust to make the flow from the two boundary layers compatible. The fluid between the two boundary layers may rotate at a rate intermediate of the rotation of the disks so that there is flow into the boundary layer on the slower rotating disk and away from the boundary layer on the faster moving disk. An exact speed of rotation that will accomplish this matching of the axial velocities outside the boundary layers when the disks are rotating in the same direction may be obtained from the numerical results of Rogers and Lance (Ref. 13). The rate of rotation of the main fluid as a function of the ratio of the rates of rotation of the two disks is given for this case in Figure 5. The problem is more complicated when the disks rotate in opposite directions since a reversal in the tangential velocity is required at some point between the two disks; in general, Rogers and Lance were unable to obtain solutions for this case.

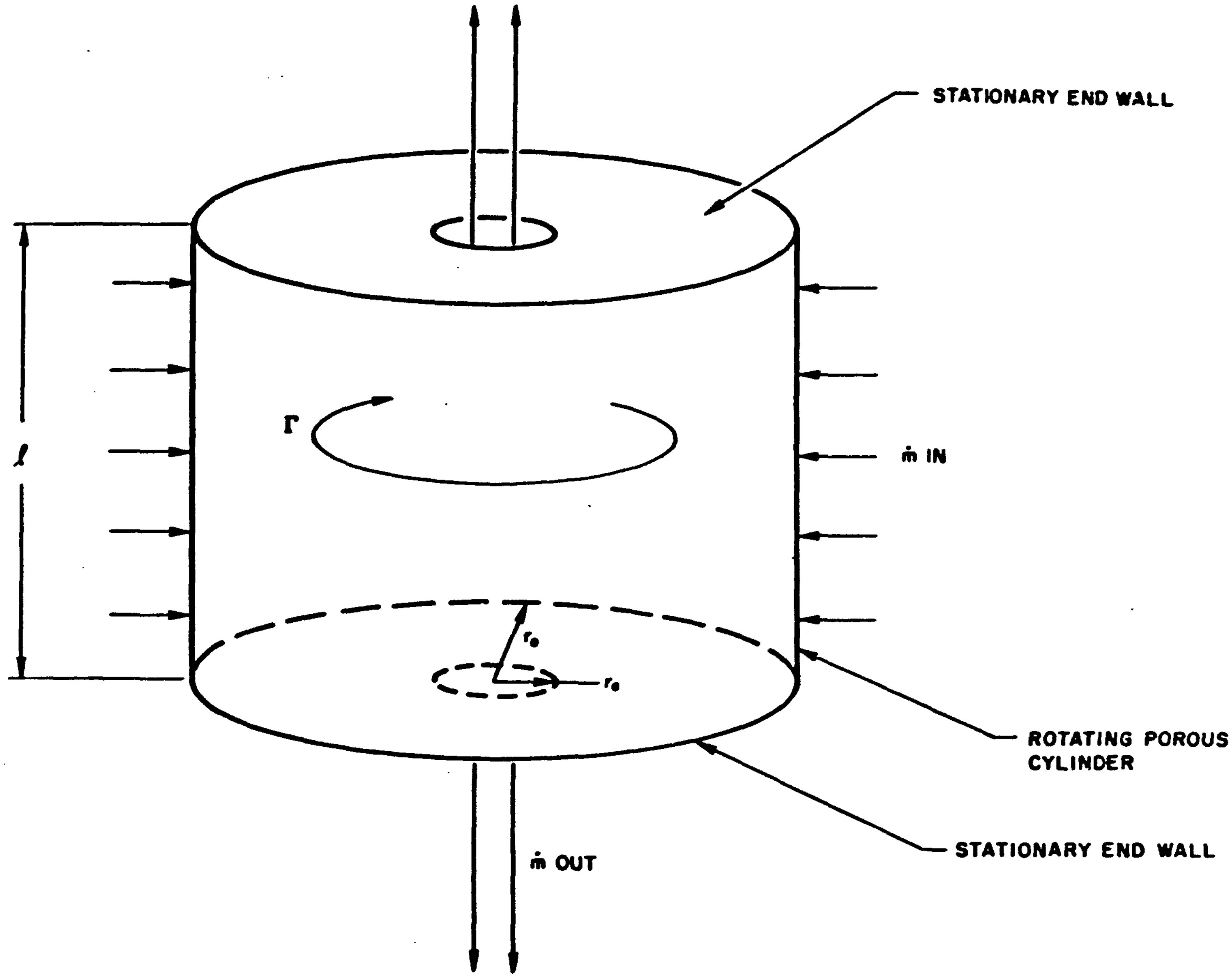

Figure 6. Sketch of the Vortex Chamber Used in the Interaction Problem

Batchelor and Stewartson both agree that the behavior shown in Figure 5 is correct when the disks are rotating at almost the same rate. However, as $\Omega_2/\Omega_1 \to 0$, Stewartson maintains that Ω_f also goes to zero. This assertion was borne out by his experiment performed with two 6-in. diameter unshrouded disks. When one disk only was rotated, the main body of air between the disks did not rotate. On the other hand, Maxworthy (Ref. 32) has performed a similar experiment, attaching to the rotating disk a cylindrical wall which encloses the fluid between the disks. The fluid on the axis between the rotating disk and the stationary disk was found to rotate at a rate close to the value predicted from the numerical results of Rogers and Lance.

In the two experiments quoted it is clear that the boundary conditions at the outer edges of the disks play a dominant role in determining the flow. This difference in the flow between the open and enclosed cases is also brought out in the work of Picha and Eckert (Ref. 33). Thus, the question is raised of whether more than one solution is possible for the limit in which the ratio of the diameter of the disks to the length between them (d/ℓ) approaches infinity. In the opinion of the present authors, the flow described by Batchelor is the only solution in this limit and the numerical results of Rogers and Lance correctly predict the flow as long as the boundary-layer thicknesses are much smaller than the distance between the disks, i.e. $(\Omega\ell^2/\nu)^{1/2} >> 1$.

A problem where the interaction is perhaps more strikingly illustrated is that of a confined vortex driven by a radial convection of angular momentum. Consider the problem of flow through a rotating porous cylinder bounded at two axial positions by stationary end walls as shown in Figure 6. Assume that radial flow enters uniformly through the porous cylinder and an axial flow leaves through exhaust holes at the center of the end walls. The angular momentum acquired by the fluid in passing through the rotating cylinder is convected in radially to set up a vortex within the container. In the regions next to the end walls, boundary-layers are formed which may have a strong influence on the vortex within the container. This problem has been

considered for laminar boundary-layer flow by Anderson (Ref. 28) and for turbulent boundary-layer flow by Rosenzweig, Lewellen, and Ross (Ref. 34). (There is also a difference in the way fluid is exhausted from the container in Refs. 28 and 34.) A corresponding linear problem for the complete container rotating has been considered by Lewellen (Ref. 21).

The character of the vortex in the primary flow is largely determined by the local Reynolds number Re based on the radial mass flow at any radius (Ref. 2). If the radial Reynolds number is large ($\gtrsim 5$), the vortex is very nearly potential ($v \sim 1/r$) except very near to the center where the tangential velocity must go to zero. If the radial Reynolds number is small ($\lesssim 1$) the vortex is very nearly in solid-body rotation ($v \sim r$). On the other hand, the strength of the vortex controls the amount of radial mass flow induced in the boundary layer, as seen in the preceding sections. By continuity, radial flow diverted to the boundary layer reduces that passing through the vortex. This then is the interaction problem: a strong vortex forces radial mass flow into the boundary layers; this diversion of radial flow to the boundary layers reduces the strength of the vortex; and the weakening vortex forces flow out of the boundary layer back into the main flow.

The strength of this interaction is determined by the ratio of the mass flow the boundary layers are potentially capable of carrying to the total mass flow passing through the container. To be specific, consider an interaction parameter A, defined as the ratio of the radial mass flow at $r = 0$ in the boundary layers on the two end walls if the vortex remains potential to the total radial mass flow $\dot{m}$ through the container. From Section III, Eq. (5-18) gives

$$A = 2k_2\sigma^{\sigma}(\nu/\Gamma_o)^{2(1-\sigma)}\frac{\Gamma_o R_o}{\dot{m}/2\pi\rho} \qquad (6-1)$$

It can be seen that the interaction parameter will remain small unless the ratio $\dot{m}/2\pi\rho\Gamma_o R_o$ is small. This ratio of radial flow to circulation, which governs the relative effects of rotation, is a Rossby number. In Refs. 5

and 21 it has been shown that when such a parameter is small it is advantageous to expand the primary flow outside the boundary layer in terms of this parameter. The leading terms in the resulting stream function ψ expansion are dependent only on the boundary conditions of the primary flow. The leading term in the circulation expansion is then determined from the equation

$$r^2 \frac{d}{dr}\left(\frac{1}{r}\frac{d\Gamma}{dr}\right) - \frac{1}{\nu l}[\psi(r, l) - \psi(r, 0)]\frac{d\Gamma}{dr} = 0 \qquad (6\text{-}2)$$

When $z = 0$ and $z = l$ are taken as the edges of the boundary layers on the two end walls, $\psi(r, l) - \psi(r, 0)$ is determined from the boundary-layer equations. In fact,

$$\frac{1}{\nu l}[\psi(r, l) - \psi(r, 0)] = -\mathrm{Re}\left\{1 - \frac{4\pi\rho\Omega}{\dot{m}}\right\} \qquad (6\text{-}3)$$

with Re defined as $\dot{m}/2\pi l\rho\nu$.

The methods of solution discussed in Section III may be used to solve for the flow in the boundary layer as a function of an arbitrary circulation distribution in the main flow. The interaction problem then is to solve Eqs. (6-2) and (6-3) coupled to the boundary-layer equation (5-18).

In general, the simultaneous solution of Eqs. (6-2), (6-3), and (5-18) requires numerical integration. However, there is an interesting analytic solution in the limit of large radial Reynolds numbers. In this limit the circulation is constant for

$$r \geq \hat{r} = r_o(1 - A^{-1/\sigma})^{\sigma} \qquad (6\text{-}4)$$

while the total mass flow is being diverted to the boundary layers. For $r \leq \hat{r}$, all of the radial flow is confined to the boundary layers, and the circulation profile assumes the distribution that will keep the radial flow

in the boundary layer constant. From Eqs. (6-2) and (6-3), it is seen that any Γ distribution is possible in the limit of $Re \to \infty$ and $Q \to \dot{m}/4\pi\rho$. From Eqs. (5-9) and (5-10), the Γ distribution that will keep $Q = \dot{m}/4\pi\rho$ is found to be

$$\Gamma = \Gamma_o \left\{ 1 - \left(\frac{2\sigma - 1}{\lambda_1 - 1} \right) A^{1/\sigma} \left[\left(\frac{r}{r_o} \right)^{1/\sigma} - \left(\frac{\hat{r}}{r_o} \right)^{1/\sigma} \right] \right\}^{\sigma/(1-2\sigma)} \tag{6-5}$$

This Γ distribution is valid for $r_e \leq r \leq \hat{r}$. Inside r_e the distribution depends upon the details of the exhaust and will not be considered here. This limiting solution is compared with numerical results in Ref. 34. There is good agreement even at a radial Reynolds number of 10, except in the immediate neighborhood of $r = \hat{r}$ and $r = r_e$ where sharp breaks occur in the infinite Reynolds number limit. Equation (6-5) gives a good approximation for the tangential velocity distribution when the viscous effects in the boundary layer are much more important than in the outer vortex flow.

Having considered two examples of the interaction occurring due to a rotating boundary layer, let us speculate on the possible importance of such an interaction effect in the case of the boundary layer associated with a tornado. A review of some mathematical models proposed for the dynamics of a tornado has been given in Ref. 21. None of the models reviewed consider the influence of the ground boundary layer.

Some of the essential flow pattern associated with a tornado appears in a model proposed by Rott (Ref. 35) in which

$$u = -ar \quad ; \quad w = 2az \tag{6-6}$$

$$v = \frac{\Gamma_\infty}{r}\left[1 - e^{-ar^2/2\nu}\right] \tag{6-7}$$

Equations (6-7) and (6-8), which represent an exact solution of the incompressible Navier-Stokes equations, yield a viscous core size of

$$r^* = \sqrt{\frac{2\nu}{a}} \tag{6-8}$$

with

$$v_{max} = 0.64 \frac{\Gamma_\infty}{r^*} \tag{6-9}$$

Although very few of its characteristics are known precisely, it does appear that for a tornado Γ_∞ is of the order of 10^4 m^2/sec and v_{max} is of the order of 10^2 m/sec (Ref. 4). If it is assumed that $a = 10^{-2}$ sec^{-1} (corresponding to a reasonable vertical velocity of 10 m/sec at z = 500 m), then for Eqs. (6-8) and (6-9) to be consistent it is necessary to have $\nu = 0(10$ m^2/sec). This value of viscosity is six orders of magnitude larger than the laminar value of air but is of the same order of magnitude as the turbulent virtual viscosity assumed by some other investigators.

Consider now the boundary-layer flow near the ground which would be induced by this model. Although it requires a rather large extrapolation, an estimate of the mass flow in the boundary layer can be obtained from the analysis in Section V of a turbulent boundary layer on a disk. The flow diverted to the ground layer at a large radius must be returned to the flow within the viscous core. The possible importance of this flow diversion to the overall tornado may be indicated by a ratio of the radial flow outside the boundary layer to that within the boundary layer at r*. Using Eq. (5-26) to estimate the boundary layer flow, one obtains

$$\left.\frac{2\pi\rho Q}{\dot{m}}\right|_{r^*} = \frac{0.13\nu^{1/5}\Gamma_\infty^{4/5} r_o}{a r^{*2} \ell} \tag{6-10}$$

where l is the axial extent of the radial inflow in the tornado. For an r_o/l of order one and the previously mentioned values for the other parameters, this ratio of mass flows is of order one.

Based on the mass-flow ratios involved, it has just been demonstrated that the influence of the boundary layer may be significant. This influence of the boundary layer may be limited to significantly increasing the vertical velocity within the viscous core of the tornado. However, a further interaction between the boundary layer and the primary tornado flow becomes important when the required sink at altitude is considered to be limited rather than infinite. Then, an interaction similar to that found in the rotating porous cylinder will occur because of the division of the flow between the boundary layer and the main flow. (For further details see Ref. 21). The extent to which the ground layer does influence the dynamics of the tornado may possibly be determined by measurements of the sink strength at altitude.

VII. CONCLUDING REMARKS

A survey of work on rotating boundary layers has been given. Perhaps the most significant extension to existing theories which has been included is the general momentum-integral solution for the boundary layer formed by a rotating fluid in contact with a stationary wall. In this solution, the boundary-layer development may be either laminar or turbulent and the wall contour rather arbitrarily orientated with respect to the axis of rotation.

Several areas open for future research are suggested by the present survey. One area which should prove fruitful is that of shear layers between rotating and nonrotating fluids. One special type of shear layer was mentioned in the discussion of Stewartson's work in Section II. Actually, shear layers analogous to all the different types of boundary layers discussed herein should be possible. The study of radial shear layers may be an advantageous way of approaching the intriguing "vortex breakdown" problem (Ref. 36).

An interesting boundary-layer question concerns the extent to which the flow solution calculated by Bödewadt is approached at the center of a finite stationary disk perpendicular to the axis of rotation of a fluid in uniform rotation. Schwiderski and Lugt (Ref. 37) have touched upon this, as have many others (Refs. 16, 17, 30, 38), but the question remains unanswered.

The boundary-layer solutions discussed herein have a wide range of applications. Probably the most undeveloped of these is in the area of ground effect on such atmospheric phenomena as cyclones, tornadoes, and dust devils.

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